Unified framework for continuous- and discrete-time Nash realizations

Jana Němcová[1] and Ewa Pawłuszewicz[2] and Mihály Petreczky[3]

***Abstract*— In this paper we investigate realization theory of a class of non-linear systems, called Nash systems, both in continuous and discrete time. To unify the approach for continuous and discrete time we will consider Nash systems on time scales. Nash systems are non-linear systems whose vector fields and readout maps are analytic semi-algebraic functions. We will present necessary and sufficient conditions for a given response map to have a realization as a Nash system on the time scale, and we will relate observability and reachability of Nash systems to minimality. The main contribution of the paper is twofold: 1) considering time scales framework for the class of semi-algebraic systems and solving realization problem, 2) results concerning discrete-time counterpart of the obtained characterization of the existence of Nash realizations on time scales.**

I. Introduction

Realization theory deals basically with three problems:

- *Existence* Under which conditions does there exist a system of a certain class generating the specified input/output behavior?
- *Properties* Under which conditions does there exist a system of the given class which is observable, reachable, minimal?
- *Algorithms* How to construct a system from the input-output behavior?

In this paper we deal only with the first two problems for the class of Nash systems both in continuous and discrete time. Nash systems are non-linear systems whose dynamics and readout maps are defined by using analytic semi-algebraic functions, i.e. analytic functions which satisfy an algebraic equation.

Our primary motivation to study Nash systems is the role they play in systems biology, see [4] for a detailed discussion. They are widely used to model metabolic, signaling and genetic networks. Moreover, since polynomial and regular rational systems are also Nash systels, Nash systems can be directly applied for other modeling approaches such as Michaelis-Menten and mass action kinetics. Further, because of the algebraic nature of Nash systems, one expects implementation of the obtained theoretical results by means of existing computer algebra packages.

Note that the continuous-time case results have already been presented in [4]. Hence the main novelty of this paper is 1) a solution to the existence problem of Nash discrete-time realizations and 2) a unified approach to realization theory of continuous-time and discrete-time Nash systems. The obtained results on the existence problem of realization theory of Nash systems on time scales may contribute to designing novel identifiability analysis procedures for biochemical models, parameter estimation techniques and reduction procedures to obtain biochemical models of lower dimensions or of properties like being observable, reachable.

We present our results in slightly simplified way to make the approach more obvious and more transparent to the reader. All technical details and general versions of the statements will be included in a journal paper of this extended abstract.

II. Framework

The notation and basic notions used in this paper are adopted from [1], [4]. For more details on used commutative algebraic and algebraic geometric concepts see [2].

A set $X \subseteq \mathbb{R}^n$ is called *semi-algebraic* if it is a set of points of $\mathbb{R}^n$ which satisfy finitely many polynomial equalities and inequalities, or if it is a finite union of such sets. Let $X \subseteq \mathbb{R}^n$, $X' \subseteq \mathbb{R}^m$ be semi-algebraic sets. A function $f : X \to X'$ is a *semi-algebraic function* if the graph of f is a semi-algebraic set in $\mathbb{R}^{n+m}$. Finally, a *Nash function* on an open semi-algebraic set X is an analytic and semi-algebraic function from X to $\mathbb{R}$. The ring of all Nash functions on X is denoted by $\mathcal{N}(X)$. If A is an integral domain, then denote by trdeg A the *transcendence degree* of A.

A. Time scales

A time scale is an arbitrary non-empty closed subset of $\mathbb{R}$ with the topology inherited from $\mathbb{R}$. However, in this paper we will consider only the following cases: $\mathbb{T} = \langle 0, +\infty)$ (continuous-time), and $\mathbb{T} = \mathbb{N}$ (discrete-time). If $c, d \in \mathbb{T}$ and $c \leq d$ then denote by $\langle c, d\rangle$ the set $\{x \in \mathbb{T} \mid c \leq x \leq d\}$.

B. Piecewise-constant inputs

Denote by $\Omega = \mathbb{R}^m$ the set of input values. We denote by U_{pc} a set of piecewise-constant input functions $u : \langle 0, d\rangle \to \Omega$, i.e. $u \in U_{pc}$ if and only if there exist $t_0 = 0 < t_1 < \ldots < t_k < t_{k+1} = d$ and $\omega_0, \ldots, \omega_k \in \Omega$ such that $u(t_0) = \omega_0$ and for all $i = 0, \ldots, k$ and for all $t \in (t_i, t_{i+1}\rangle$, $u(t) = \omega_i$.

*This work was partially supported by the GAČR project 13-16764P, by Bialystok University of Technology grant S/WM/2/2012 and by ESTIREZ project of Région Nord-Pas de Calais, France.

[1]Jana Němcová is with the Department of Mathematics, Institute of Chemical Technology, Technická 5, 166 28 Prague 6, Czech republic Jana.Nemcova@vscht.cz

[2]Ewa Pawłuszewicz is with the Faculty of Mechanical Engineering, Białystok University of Technology, Wieska 45C, 15-351 Bialystok, Poland epaw@pb.bialystok.pl

[3]Mihály Petreczky is with the Department of Computer Science and Automatic Control, Ecole des Mines de Douai, 941, rue Charles Bourseul, 59508 Douai, France mihaly.petreczky@mines-douai.fr

For a constant input $u \in U_{pc}$ with the value $\omega \in \Omega$ on the time interval $\langle 0, d\rangle \subseteq \mathbb{T}$ ($u(t) = \omega$ for all $t \in \langle 0, d\rangle$) we use the notation $u = [\omega, d]$. The final time of u is denoted by T_u. Hence, if u is the constant input $[\omega, d]$, then $T_u = d$. If $\langle a, b\rangle \subseteq \langle 0, d\rangle$, then denote by $u|_{\langle a,b\rangle}$ the restriction of u to the set $\langle a, b\rangle$.

For $u, v \in U_{pc}$ one can define concatenation $u \sqcup v$ as follows:

$$(u \sqcup v)(t) = \begin{cases} u(t) & t \in \langle 0, T_u\rangle \\ v(t - T_u) & t \in (T_u, T_u + T_v\rangle \end{cases}$$

By the empty input $e = [\omega, T_e]$ on $\mathbb{T}$ we refer to the input e such that $0 = T_e$ in case $\mathbb{T} = \langle 0, \infty)$ or such that $\omega = 0$ in case $\mathbb{T} = \mathbb{N}$. By convention, for any $u \in U_{pc}$, $u \sqcup e = e \sqcup u = u$.

Hence, every $u \in U_{pc}$ can be written in the form $u = [\omega_1, t_1] \sqcup \cdots \sqcup [\omega_k, t_k]$ which means that $u(t) = \omega_i$ for $t \in (t_{i-1}, t_i\rangle$ and $u(0) = \omega_1$. The set of all piecewise-constant inputs of U_{pc} with exactly k constant pieces is denoted by U^k_{pc}. Hence, every input u of U^k_{pc} has k switches (the control values of u change k-times).

C. Nash systems on time scales

Definition 2.1: Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. The *delta derivative* of f at t, denoted by $f^\Delta(t)$ or $\frac{\Delta}{\Delta t} f(t)$, is the real number (provided it exists) such that $\forall \epsilon > 0 \exists \delta > 0 \forall s \in \{\tau \in \mathbb{T} \mid t - \delta < \tau < t + \delta\}$:

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

where $\sigma : \mathbb{T} \to \mathbb{T}$ is so-called *forward jump operator* defined as $\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}$. We say that f is delta differentiable on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}$. Sometimes,we will denote by $\frac{\Delta}{\Delta t} f(t)|_{t=\tau}$ the delta derivative $f^\Delta(\tau)$.

Note that in continuous time $\frac{\Delta}{\Delta t}$ corresponds to the standard differentiation, while in discrete-time it denotes the forward difference.

Theorem 2.2: [3] Let $x : \mathbb{T} \to \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be delta differentiable. Then

$$\frac{\Delta}{\Delta\, t}\varphi(x(t)) = \int_0^1 [\varphi'(x(t) + h\mu(t)x^\Delta(t))]dh \cdot x^\Delta(t),$$

where φ' denotes the standard gradient of φ and $\mu(t)$ denotes the so-called *graininess function* at t defined by $\mu(t) = \sigma(t) - t$.

Definition 2.3: A Nash system Σ with the input space $\Omega = \mathbb{R}^m$ and an output space $\mathbb{R}^r$ on a time scale $\mathbb{T}$ is a quadruple (X, f, h, x_0) such that:

1) *state space:* X is an open connected semi-algebraic subset of $\mathbb{R}^n$
2) *dynamics:* $x^\Delta(t) = f(x(t), u(t))$, where $t \in \mathbb{T}$, $U_{pc} \ni u : \mathbb{T} \to \Omega$ and $f : X \times \Omega \to \mathbb{R}^n$ is such that for every control value $\omega \in \Omega$ of u the function $f_\omega : X \ni x \mapsto f(x, \omega) \in \mathbb{R}^n$ is a Nash function
3) *output function:* $y(t) = h(x(t))$, where $h : X \to \mathbb{R}^r$ is a Nash function
4) *initial state:* $x_0 = x(0) \in X$

In addition, we assume that in the discrete-time case, i.e. when $\mathbb{T} = \mathbb{N}$, $f(x, 0) = 0$, $f(x, \omega)$ is analytic in ω and for every $\omega \in \Omega$, $x \in X$, $f(x, \omega) + x \in X$.

Denote by $\dim \Sigma$ the dimension n of the state-space of Σ.

Remark 2.4: The assumption that $f(x, 0) = 0$ for discrete-time Nash systems can be replaced by requiring that $\psi(T_{\varphi(u,0)}, x, \varphi(u, 0)) = x$ holds for both continuous- and discrete-time cases, where ψ is the solution function of the system defined below. It is easy to see that this holds automatically for continuous-time case and amounts to $f(x, 0) = 0$ for the discrete-time case. The assumption that $f(x, \omega) + x \in X$ can be replaced by the requirement that for every $x \in X$, exists $\epsilon > 0$ such that the flow $\psi(., x, [\omega, \epsilon])$ is well defined and takes values in X.

The trajectory of $\Sigma = (X, f, h, x_0)$ corresponding to an input $U_{pc} \ni u = [\omega_1, t_1] \sqcup [\omega_2, t_2] \sqcup \cdots \sqcup [\omega_k, t_k] : \langle 0, T_u := \sum_{i=1}^k t_i\rangle \to \Omega$ is a function $x(\cdot) = \psi(\cdot, x_0, u) : \langle 0, T_u\rangle \to X$ defined as:

- $x(0) = \psi(0, x_0, u) = x_0$,
- restriction of $x(t)$ to the interval $\langle c, d\rangle = \langle \sum_{j=1}^{i-1} t_j, \sum_{j=1}^{i} t_j\rangle$ is the unique solution to the initial problem $x^\Delta(t) = f_{\omega_i}(x(t))$, $x(c) = \psi(c, x_0, [\omega_1, t_1] \sqcup \cdots \sqcup [\omega_{i-1}, t_{i-1}])$, provided $x(t) \in X$ is defined for all $t \in \langle c, d\rangle$.

Note that the trajectory of $\Sigma = (X, f, h, x_0)$ does not need to exist for every $u \in U_{pc}$. We will call the controls $u \in U_{pc}$ for which the trajectory exists *admissible* and we will use the notation $\mathscr{U}_\Sigma$ for the set of all such controls.

Assumption 2.5: In the rest of the paper we will consider a subset $\mathscr{U}_{pc} \subseteq U_{pc}$ of piecewise-constant control, such that

- $\forall \omega \in \Omega\ \exists t \in \mathbb{T}:\ [\omega, t] \in \mathscr{U}_{pc}$
- $\forall u \in \mathscr{U}_{pc}\ \forall t \in \langle 0, T_u\rangle:\ u|_{\langle 0,t\rangle} \in \mathscr{U}_{pc}$
- $\forall u \in \mathscr{U}_{pc}\ \forall \omega \in \Omega\ \exists \epsilon > 0:\ T_u + \epsilon \in \mathbb{T}$ and $u \sqcup [\omega, \epsilon] \in \mathscr{U}_{pc}$

Note that this assumption is not restrictive once admissible controls are considered. Namely, first two properties are always satisfied and the third one refers to a kind of local invariance of a state-space (it always holds in case of $\mathbb{R}^n$ state-space).

III. Realization problem

For every Nash system $\Sigma = (X, f, h, x_0)$ one can define a response map which assigns to an input $u \in \mathscr{U}_\Sigma$ the value $h(\psi(T_u, x_0, u)) \in \mathbb{R}^r$. The problem we deal with in this paper is whether for a given response map p one can find a corresponding Nash system. We call such system *realization of* p. Let us introduce the right formalization of response maps to deal with Nash realizations on time scales.

Definition 3.1: Denote by $\mathscr{U}^k_{pc}$ the set of all piecewise-constant inputs from $\mathscr{U}_{pc}$ which can be represented as a concatenation of exactly k constant inputs, i.e. $u \in \mathscr{U}^k_{pc}$ if and only if $u \in \mathscr{U}_{pc}$ and there exist constant inputs $[\omega_i, t_i]$, $i = 1, \ldots, k$ such that $u = [\omega_1, t_1] \sqcup \cdots \sqcup [\omega_k, t_k]$.

Let us define a map $\varphi : \mathscr{U}^k_{pc} \times \langle 0, +\infty)^k \to \mathscr{U}^k_{pc}$ for arbitrary $k \in \mathbb{N}$, in the following way:
Let $u = [\omega_1, t_1] \sqcup \cdots \sqcup [\omega_k, t_k] \in \mathscr{U}^k_{pc}$ be arbitrary. Then,

$$\varphi(u, \alpha_1, \ldots, \alpha_k) := [\kappa_1, \delta_1] \sqcup \cdots \sqcup [\kappa_k, \delta_k],$$

where

- *in case of continuous time scale:*

$$[\kappa_i, \delta_i] = [\omega_i, \alpha_i t_i] \text{ for } i = 1, \ldots, k.$$

- *in case of discrete time scale:*

$$[\kappa_i, \delta_i] = \begin{cases} [\alpha_i\omega_i, t_i] & \text{if } \alpha_i > 0 \\ e & \text{if } \alpha_i = 0 \end{cases} \quad \text{for } i = 1, \ldots, k.$$

It is possible to define φ in more general way by specifying number of properties it has to satisfy. However, for the purposes of this extended abstract the definition above is sufficient.

Remark 3.2: Note that if $u \in \mathscr{U}^k_{pc}$, $v \in \mathscr{U}^l_{pc}$ and $\alpha \in \mathbb{R}^k$, $\beta \in \mathbb{R}^l$ then $\varphi(u \sqcup v, \alpha, \beta) = \varphi(u, \alpha) \sqcup \varphi(v, \beta)$.

Definition 3.3: We say that a function $p : \mathscr{U}_{pc} \to \mathbb{R}^r$ is a *response map* if there exists a map φ as in Definition 3.1 such that for every $k \in \mathbb{N}$ and for every $u \in \mathscr{U}^k_{pc}$ the map $p(\varphi(u, \alpha_1, \ldots, \alpha_k))$ is analytic in $\alpha_1, \ldots, \alpha_k$. The set of all response maps from $\mathscr{U}_{pc}$ to $\mathbb{R}^r$ is denoted by $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R}^r)$.

Recall that in the definition above we assume $\mathscr{U}_{pc}$ satisfies Assumption 2.5.

The existence part of the realization problem for the class of Nash systems on time scales we deal with in this paper is stated as follows:

Problem 3.4: Consider a response map $p \in \mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R}^r)$. Find a Nash system $\Sigma = (X, f, h, x_0)$, if it exists, such that

$$p(u) = h(\psi(T_u, x_0, u)) \text{ for all } u \in \mathscr{U}_{pc} \subseteq \mathscr{U}_\Sigma.$$

If the condition above holds, then we say that Σ is a realization of p.

Note that if Σ is a Nash system, then $\mathscr{U}_{pc} := \mathscr{U}_\Sigma$ is an admissible set of inputs and if we define $p_\Sigma : \mathscr{U}_{pc} \to \mathbb{R}^r$ by $p_\Sigma(u) = h(\psi(T_u, x_0, u))$ for all $u \in \mathscr{U}_{pc}$ then it is easy to see that p_Σ is a response map and Σ is a realization of p_Σ.

IV. Main results

Prior to the statement of the main results of this paper, let us recall necessary notation adopted from [1], [4].

Consider $\xi : \mathscr{U}_{pc} \to \mathbb{R}$ and $\omega \in \Omega$ and let us define the operator $\triangle_\omega$ by

$$(\triangle_\omega\, \xi)(u) = \frac{\triangle}{\triangle\, t}\xi(u \sqcup [\omega, t])|_{t=0+}$$

for any $u \in \mathscr{U}_{pc}$ and $t > 0$. Note that in the continuous-time case $(\triangle_\omega\, \xi)(u) = \frac{d}{dt}\xi(u \sqcup [\omega, t])|_{t=0+}$ and in the discrete-time case $(\triangle_\omega\, \xi)(u) = \xi(u \sqcup [\omega, 1])$.

Let X be an open connected semi-algebraic set, let $\mathscr{U}_{pc}$ be a set of piecewise-constant inputs satisfying Assumption 2.5, and let $\mathcal{A} = \{\xi_1, \ldots, \xi_n\}$ be a subset of $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$. Assume that $\forall u \in \mathscr{U}_{pc}$, $(\xi_1(u), \ldots, \xi_n(u)) \in X$. The *Nash extension* $\mathcal{A}^{Nash}(X)$ of $\mathcal{A}$ with respect to X is the subalgebra of $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ generated by the maps $\mathscr{U}_{pc} \ni u \mapsto q(\xi_1(u), \ldots, \xi_n(u)) \in \mathbb{R}$, where $q \in \mathcal{N}(X)$. Note that $\mathcal{A}^{Nash}(X)$ can be thought of as the generalization of the notions of linear space and algebra generated by $\mathcal{A}$.

Theorem 4.1: $p : \mathscr{U}_{pc} \to \mathbb{R}^r$ has a Nash realization if and only if there exist a set $\mathcal{A} = \{\xi_1, \ldots, \xi_n\} \subseteq \mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ and an open connected semi-algebraic set $X \subseteq \mathbb{R}^n$ such that:

1) $\forall u \in \mathscr{U}_{pc} : (\xi_1(u), \ldots, \xi_n(u)) \in X$
2) $\forall i = 1, \ldots, r : p_i \in \mathcal{A}^{Nash}(X)$
3) $\forall \xi \in \mathcal{A}^{Nash}(X) \forall \omega \in \Omega : \triangle_\omega\, \xi \in \mathcal{A}^{Nash}(X)$

This theorem is the generalization of [4, Theorem 5.5] which was stated only for continuous-time Nash realizations. One can also view it as a generalization of [1, Theorem 4.6] since in [1] the results are stated for the realizations on time scales but for realizations within different class of systems.

Consider now the set $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ of scalar valued response maps. It is easy to see that $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ is a commutative algebra over $\mathbb{R}$ under the operations of pointwise addition and multiplication: its zero element is the constant zero map and its unit is the constant 1 map. Let $p : \mathscr{U}_{pc} \to \mathbb{R}^r$ be a response map and let $p_1, \ldots, p_r : \mathscr{U}_{pc} \to \mathbb{R}$ be the coordinates of p: $p(u) = (p_1(u), \ldots, p_r(u))$ for all $u \in \mathscr{U}_{pc}$. Then $A_{obs}(p)$ denotes the smallest subalgebra of $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ which is closed with respect to the operators $\triangle_\omega$, $\omega \in \Omega$ and which contains $p_1, \ldots, p_r$, i.e. $A_{obs}(p) =: A$ is the smallest subalgebra of $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ such that $\forall g \in A\, \forall \omega \in \Omega : \triangle_\omega\, g \in A$ and $p_1, \ldots, p_r \in A$. In analogy to dimension of linear spaces, one would like to compute the maximal number of algebraically independent elements to specify the transcendence degree (denoted trdeg) of $A_{obs}(p)$. However, to be able to compute trdeg $A_{obs}(p)$ it is crucial for $A_{obs}(p)$ to be an integral domain. This follows from the following theorem.

Theorem 4.2: $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ is an integral domain.

We can now state the following necessary condition for existence of a Nash realization.

Theorem 4.3: Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \mathscr{U}_{pc} \to \mathbb{R}^r$. Then trdeg $A_{obs}(p) \leq \dim \Sigma < +\infty$.

Finally, we relate minimality of Nash realizations to their reachability and observability. First, let us define these notions. The respective definitions are introduced in [4] for continuous-time Nash systems.

Definition 4.4: Let $p : \mathscr{U}_{pc} \to \mathbb{R}^r$ be a response map and let $\Sigma = (X, f, h, x_0)$ be a Nash realization of p. We denote by $\mathcal{R}(x_0)$ the set of states of Σ reachable from x_0 by the inputs of $\mathscr{U}_{pc}$, i.e.

$$\mathcal{R}(x_0) = \{\psi(T_u, x_0, u) \mid u \in \mathscr{U}_{pc}\}.$$

We say that $\Sigma = (X, f, h, x_0)$ is *semi-algebraically reachable*, if no non-zero element of $\mathcal{N}(X)$ vanishes on the set of states of Σ reachable from x_0, i.e.

$$\forall g \in \mathcal{N}(X) : ((g = 0 \text{ on } \mathcal{R}(x_0)) \Rightarrow g = 0)\,.$$

Definition 4.5: The *observation algebra* $A_{obs}(\Sigma)$ of a Nash system $\Sigma = (X, f, h, x_0)$ is the smallest subalgebra of $\mathcal{N}(X)$ which contains h_i, $i = 1, \ldots, r$ and such that if $g \in A_{obs}(\Sigma)$, then the function $L_\omega g : X \ni x \mapsto \frac{\triangle}{\triangle t}g(\psi(t, x, [\omega, t]))|_{t=0+}$ also belongs to $A_{obs}(\Sigma)$. We call Σ *semi-algebraically observable*, if trdeg $A_{obs}(\Sigma) = \dim \Sigma$.

In continuous time, the function $L_\omega g$ is the Lie-derivative of g with respect to the vector field $f_\omega : X \ni x \mapsto f(x, \omega)$, in the discrete-time case, $L_\omega g(x) = g(f_\omega(x) + x) - g(x)$. Recall that the ring of Nash functions $\mathcal{N}(X)$ is an integral domain, and hence so is $A_{obs}(\Sigma)$, and thus the transcendence degree of $A_{obs}(\Sigma)$ is well defined.

Definition 4.6: We say that a Nash realization $\Sigma = (X, f, h, x_0)$ of a response map p is a *minimal* Nash realization of p if for any Nash realization Σ' of p it holds that $\dim \Sigma \leq \dim \Sigma'$.

The following three theorems describing the relation of minimality, semi-algebraic reachability and semi-algebraic observability of Nash realizations on time scales are the generalizations of the respective theorems in [4].

Theorem 4.7: If the dimension of a Nash realization Σ of a response map p equals trdeg $A_{obs}(p)$, then Σ is a minimal Nash realization of p.

Theorem 4.8: A Nash realization Σ of a response map p is semi-algebraically reachable and semi-algebraically observable if and only if $\dim(\Sigma) =$ trdeg $A_{obs}(p)$.

Theorem 4.9: If a Nash realization Σ of a response map p is semi-algebraically reachable and semi-algebraically observable, then Σ is minimal.

V. PROOFS

To prove Theorem 4.1 one proceeds in the same way as in the proofs of [1, Theorem 4.6] and [4, Theorem 5.5].
Let $\Sigma = (X, f, h, x_0)$ be a realization of p. One can show that the set $\mathcal{A} = \{\xi_1, \dots, \xi_n\}$, where $\xi_i \in \mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ is defined as $\xi_i(u) = \psi(T_u, x_0, u)$, i.e. as the i-th component of the state of the system Σ at the time T_u under the input $u \in \mathscr{U}_{pc}$, satisfies the conditions of the theorem. Recall that $\psi(\cdot, x_0, u)$ refers to the trajectory of Σ corresponding to the input u.
For the other implication, one constructs a realization of p from $\mathcal{A} = \{\xi_1, \dots, \xi_n\} \subseteq \mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ and $X \subseteq \mathbb{R}^n$ as in the theorem. First, according to the conditions 1), 2) of the theorem one derives $h_i, f_{\omega,j} \in \mathcal{N}(X)$ such that

$$p_i(u) = h_i(\xi_1(u), \dots, \xi_n(u)),$$

$$(\triangle_\omega \, \xi_j)(u) = f_{\omega,j}(\xi_1(u), \dots, \xi_n(u)),$$

for $i = 1, \dots, r$, $j = 1, \dots, n$, $\omega \in \Omega$, $u \in \mathscr{U}_{pc}$. Then the system $\Sigma = (X, f, h, x_0)$, where

- $f : X \times \Omega \to \mathbb{R}^n$ is such that for every control value $\omega \in \Omega$ the j-th component of the function $f(\cdot, \omega)$ is given by $f_{\omega,j}(\cdot)$,
- the i-th component of h is defined as h_i,
- the initial state is $x_0 = (\xi_1(e), \dots, \xi_n(e))$, where e is the empty input,

is a realization of p.
■

Theorem 4.9 is a corollary of Theorems 4.7 and 4.8.
■

In order to prove Theorems 4.3, 4.7, 4.8, we need to define the map τ_Σ^* for any Nash system $\Sigma = (X, f, h, x_0)$ which is a realization of a response map $p \in \mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R}^r)$. Let $\tau_\Sigma^* : \mathcal{N}(X) \to \mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ be defined as follows:

$$\forall g \in \mathcal{N}(X) \, \forall u \in \mathscr{U}_{pc} : \tau_\Sigma^*(g)(u) = g(\psi(T_u, x_0, u)).$$

By an argument similar to the one presented in [4], it can be shown that $\tau_\Sigma^*(A_{obs}(\Sigma)) = A_{obs}(p)$, τ_Σ^* is injective if and only if Σ is semi-algebraically reachable. Then the proofs of Theorems 4.3, 4.7, 4.8 can be done in a way which is analogous to what was presented in [4]. For example, in order to prove Theorem 4.3, it is sufficient to notice that the ring of Nash functions $\mathcal{N}(X)$ is algebraic over polynomials $\mathbb{R}[X]$ on X and thus trdeg $\mathcal{N}(X) =$ trdeg $\mathbb{R}[X] = \dim \Sigma$, and that trdeg $A_{obs}(p) =$ trdeg $\tau_\Sigma^*(A_{obs}(\Sigma)) \leq$ trdeg $\mathcal{N}(X)$. In order to prove Theorem 4.8, it suffices to notice that if Σ is semi-algebraically reachable then τ_Σ^* is injective and hence trdeg $A_{obs}(p) =$ trdeg $\tau_\Sigma^*(A_{obs}(\Sigma)) =$ trdeg $A_{obs}(\Sigma) =$ trdeg $\mathcal{N}(X) = \dim \Sigma$.
■

The reasoning above relies in an essential way on Theorem 4.2. Due to its complexity, we omit the proof of this theorem. Intuitively, the proof relies on the fact that for any response map p and any input $u \in \mathscr{U}_{pc}^k$, $p(\varphi(u, \alpha_1, \dots, \alpha_k))$ is analytic in $\alpha_1, \dots, \alpha_k$. Then the fact that $\mathcal{A}(\mathscr{U}_{pc} \to \mathbb{R})$ is an integral domain is inherited from the fact that the ring of analytic functions defined on a connected set is an integral domain. In order to ensure that we can speak of connected sets, we needed the property that $f(x, 0) = 0$ in the discrete-time case.
■

VI. CONCLUSIONS

We provided two characterizations of the existence of Nash realizations on time scales of a given response map and we specified the relations between their minimality, observability and reachability. The framework of time scales allowed us to formulate the results for continuous and discrete time in unified way.

Further research is needed to derive specific computer algebra methods for respective computations. The results will be also useful for model reduction and system identification of models within the class of Nash systems either in continuous or discrete time.

REFERENCES

[1] Z. Bartosiewicz, E. Pawluszewicz, Realizations of nonlinear control systems on time scales, IEEE Transactions on Automatic Control 53, pp. 571-575, 2008.

[2] J. Bochnak, M. Coste, M.-F. Roy *Real Algebraic Geometry*, Springer-Verlag, Berlin, 1998.

[3] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales*, Boston, Birkhauser, 2001.

[4] J. Němcová, M. Petreczky, J.H. van Schuppen, Realization theory of Nash systems, SIAM Journal on Control and Optimization, 51(5), pp. 3386-3414, 2013.