Moment Matching Based Model Reduction for LPV State-Space Models

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***Abstract*— We present a novel algorithm for reducing the state dimension, i.e., order, of linear parameter varying (LPV) discrete-time state-space (SS) models with affine dependence on the scheduling variable. The input-output behavior of the reduced order model approximates that of the original model. In fact, for input and scheduling sequences of a certain length, the input-output behaviors of the reduced and original model coincide. The proposed method can also be interpreted as a reachability and observability reduction (minimization) procedure for LPV-SS representations with affine dependence.**

I. INTRODUCTION

In control applications, it is often desirable, see [17], [15], to use discrete-time linear parameter-varying state-space representations with affine dependence on parameters (abbreviated as *LPV-SS representations* in the sequel) of the form:

$$\Sigma \begin{cases} x(t+1) = A(p(t))x(t) + B(p(t))u(t) \\ y(t) = C(p(t))x(t), \end{cases} \tag{1}$$

where $t \in \mathbb{N}$, $\mathbb{N}$ denotes the set of natural numbers including zero, $x(t) \in \mathbb{R}^{n_x}$ is the state, $y(t) \in \mathbb{R}^{n_y}$ is the output, $u(t) \in \mathbb{R}^{n_u}$ is the input, and $p(t) = \begin{bmatrix} p_1(t) & \cdots & p_{n_p}(t) \end{bmatrix}^{\mathrm{T}} \in P \subseteq \mathbb{R}^{n_p}$ is the scheduling signal at time $t \in \mathbb{N}$. Here P is an arbitrary but fixed, closed, simply connected subset of $\mathbb{R}^{n_p}$. The matrix functions $A(p(t))$, $B(p(t))$, $C(p(t))$ in (1) are assumed to be affine and static functions of $p(t)$ of the form:

$$\begin{aligned} &A(p(t)) = A_0 + \sum_{i=1}^{n_p} A_i p_i(t),\ B(p(t)) = B_0 + \sum_{i=1}^{n_p} B_i p_i(t), \\ &C(p(t)) = C_0 + \sum_{i=1}^{n_p} C_i p_i(t), \end{aligned} \tag{2}$$

where $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n_x}$ are constant matrices for all $i \in \{0, 1, \ldots, n_p\}$.

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Contribution of the paper Consider a LPV-SS representation Σ of the form (1) and fix a positive integer N. In this paper, we present a procedure for computing another LPV-SS representation

$$\bar{\Sigma} \begin{cases} \bar{x}(t+1) = \bar{A}(p(t))\bar{x}(t) + \bar{B}(p(t))u(t) \\ \bar{y}(t) = \bar{C}(p(t))\bar{x}(t), \end{cases} \tag{3}$$

such that for $x(0) = 0$, $y(t) = \bar{y}(t)$ for $0 \leq t \leq N$, for all scheduling sequences $(p(0), p(1), \ldots, p(N)) \in P^{N+1}$ and input sequences $u = (u(0), u(1), \ldots, u(N)) \in (\mathbb{R}^{n_u})^{N+1}$. Moreover, the state space dimension of $\bar{\Sigma}$ is smaller than or equal to the state space dimension of Σ. In other words, given an LPV-SS representation Σ of order n_x (state space dimension n_x) and a $N \in \mathbb{N} \backslash \{0\}$, we would like to find another LPV-SS representation $\bar{\Sigma}$ of order $r \leq n_x$ which has the same input-output behavior for all scheduling and input sequences of length up to $N+1$[1]. In addition, we would like the representation $\bar{\Sigma}$ to be a "good" approximation of Σ in terms of input-output behavior, even for scheduling and input sequences of length greater than $N+1$ (see Remark 1 for what is meant by "good" here). Intuitively, it is clear that there is a relationship between N and r: larger N yields a better approximation of the original input-output behavior, but it also results in larger value of r. In this paper, this relationship will be made more precise. Finally, by making use of this relation, the number N can be *guaranteed* to be chosen such that the resulting representation is a complete realization of the original model and it is reachable and/or observable. Therefore, the procedure stated in the present paper can also be used for reachability or observability reduction (hence, minimization) of an LPV-SS representation.

Motivation LPV-SS representations are used in a wide variety of applications, see for instance [11], [20], [5], [19], [6]. Their popularity is due to their ability to capture nonlinear dynamics, while remaining simple enough to allow effective control synthesis, for example, by using optimal $\mathscr{H}_2/\mathscr{H}_\infty$ control, MPC or PID approaches. LPV-SS representations arising in practice, especially originating from first-principles based modeling, often have a large number of states. This is due to the inherent complexity of the physical processes whose behavior the LPV-SS representations are supposed to capture. Unfortunately, due to memory limitations and numerical issues, the existing LPV controller synthesis tools are not always capable of handling large state-space representations [9]. Moreover, even if the control synthesis is suc-

[1]Note that finding a representation $\bar{\Sigma}$ with the same number of states as Σ is in fact not necessarily useful, but it can happen that the proposed method does not allow us any other option.

cessful, large plant models lead to large controllers. In turn, large controllers are more difficult and costly to implement, and they often require application of reduction techniques. For this reason, model reduction of LPV-SS representations is extremely relevant for improving the applicability of LPV systems.

Related work The tools which have been used in this paper stem from realization theory of LPV-SS representations [13], [18]. Similar tools were used for linear switched systems in [3]. In fact, we use the relationship between LPV-SS representations and linear switched systems derived in [13] to adapt the tools of [3] to LPV-SS representations. The method employed in this paper is related to that of [18]. The main difference is that [18] requires the explicit computation of Hankel matrices of LPV-SS representations. It should be noted that the size of the partial Hankel matrix of an LPV-SS representation increases exponentially. In contrast, the algorithm proposed in this paper does not require the explicit computation of Hankel matrices, and its worst-case computational complexity is polynomial.

Model reduction problem of LPV-SS representations was investigated in several papers [7], [8], [1], [22], [21], but except [21] they are only applicable to quadratically stable LPV systems. The method of [21] is applicable to quadratically stabilizable and detectable LPV-SS representations. In contrast, this paper does not impose any restrictions on the class of LPV-SS representations. In [16] joint reduction of the number of states and the number of scheduling parameters has been investigated. However, the method of [16] requires constructing the Hankel matrix explicitly. Hence, it suffers from the same curse of dimensionality as [18].

Outline: In Section II, we present the formal definition and main properties of LPV-SS representations. In Section III, we recall the concept of sub-Markov parameters for LPV-SS representations and give the precise problem statement. In Section IV, we present the moment matching algorithm. In Section V the algorithm is illustrated on a numerical example and its performance is compared with the one of [18].

II. DISCRETE-TIME LPV-SS REPRESENTATIONS

In this section, we present the formal definition of discrete-time LPV-SS representations and recall a number of relevant definitions. We follow the presentation of [13].

In the sequel, we will use

$$\Sigma = (n_y, n_u, n_x, \{(A_i, B_i, C_i)\}_{i=0}^{n_p}), \tag{4}$$

or simply Σ to denote a discrete-time LPV-SS representation of the form (1). In addition, we use $\mathbb{I}_{s_1}^{s_2}$ to denote the set $\mathbb{I}_{s_1}^{s_2} = \{s \in \mathbb{N} \mid s_1 \leq s \leq s_2\}$. An LPV-SS representation Σ is driven by the *free variables (inputs)* $\{u(k)\}_{k=0}^{\infty}$ and the *scheduling sequence* $\{p(k)\}_{k=0}^{\infty}$. In the sequel, regarding state trajectories, the initial state $x(0)$ for an LPV-SS representation is taken to be zero unless stated otherwise. This assumption is made to simplify notation. Note that the results of the paper can easily be extended for the case of non-zero initial state.

Notation 1: We will use $H^{\mathbb{N}}$ to denote the set of all maps of the form $f: \mathbb{N} \to H$ where H is a (possibly infinite) set. Using this, the sets $\mathscr{U}$, $\mathscr{P}$, $\mathscr{Y}$ and $\mathscr{X}$ are defined as $\mathscr{U} = U^{\mathbb{N}}$, $\mathscr{P} = P^{\mathbb{N}}$, $\mathscr{Y} = Y^{\mathbb{N}}$ and $\mathscr{X} = X^{\mathbb{N}}$ where $U = \mathbb{R}^{n_u}$, $P \subseteq \mathbb{R}^{n_p}$, $Y = \mathbb{R}^{n_y}$ and $X = \mathbb{R}^{n_x}$.

Consider an initial state $x_0 \in \mathbb{R}^{n_x}$ of the LPV-SS representation Σ of the form (1). The *input-to-state map* $X_{\Sigma,x_0}: \mathscr{U} \times \mathscr{P} \to \mathscr{X}$ and *input-output map* $Y_{\Sigma,x_0}: \mathscr{U} \times \mathscr{P} \to \mathscr{Y}$ of Σ corresponding to this initial state x_0 are defined as follows: for all sequences $\mathbf{u} = \{u(k)\}_{k=0}^{\infty} \in \mathscr{U}$ and $\mathbf{p} = \{p(k)\}_{k=0}^{\infty} \in \mathscr{P}$, let $X_{\Sigma,x_0}(\mathbf{u},\mathbf{p})(t) = x(t)$ and $Y_{\Sigma,x_0}(\mathbf{u},\mathbf{p})(t) = y(t)$, $t \in \mathbb{N}$, where $x(t)$, $y(t)$ satisfy (1) and $x(0) = x_0$. We say that Σ is *reachable*, if $\mathbb{R}^{n_x} = \operatorname{span}\{X_{\Sigma}(\mathbf{u},\mathbf{p})(t) \mid (\mathbf{u},\mathbf{p}) \in \mathscr{U} \times \mathscr{P}, t \in \mathbb{N}\}$, i.e., $\mathbb{R}^{n_x}$ is the smallest vector space containing all the states which are reachable from $x(0) = 0$ by some scheduling sequence and input sequence at some time instance t, where $t \in \mathbb{N}$. We say that Σ is *observable* if for any two initial states $x_1, x_2 \in \mathbb{R}^{n_x}$, $Y_{\Sigma,x_1} = Y_{\Sigma,x_2}$ implies $x_1 = x_2$. That is, if any two distinct initial states of an observable Σ are chosen, then for *some* input and scheduling sequence, the resulting outputs will be different. In the sequel, to simplify the notation, we will be dealing with those input-output maps of LPV-SS representations which correspond to the *zero initial state*. We will use X_Σ and Y_Σ to denote $X_{\Sigma,0}$ and $Y_{\Sigma,0}$ respectively.

The definition above implies that the potential input-output behavior of an LPV-SS representation can be formalized as a map

$$f: \mathscr{U} \times \mathscr{P} \to \mathscr{Y}. \tag{5}$$

The value $f(\mathbf{u},\mathbf{p})(t)$ represents the output of the underlying black-box system at time t, if the initial state $x(0) = 0$, the input $\mathbf{u} = \{u(k)\}_{k=0}^{\infty}$ and the scheduling sequence $\mathbf{p} = \{p(k)\}_{k=0}^{\infty}$ are fed to the system. Next, we define when an LPV-SS representation realizes (describes) f. The LPV-SS representation Σ of the form (1) is a *realization* of a map f of the form (5), if f equals the input-output map of Σ, i.e., $f = Y_\Sigma$. Two LPV-SS representations Σ_1 and Σ_2 are said to be *input-output equivalent* if $Y_{\Sigma_1} = Y_{\Sigma_2}$. Let Σ be an LPV-SS representation of the form (1).

Consider an LPV-SS representation Σ_1 of the form (1) and an LPV-SS representation Σ_2 of the form

$$\Sigma_2 = (n_y, n_u, n_x, \{(A_i^a, B_i^a, C_i^a)\}_{i=0}^{n_p}).$$

A nonsingular matrix $\mathscr{S} \in \mathbb{R}^{n_x \times n_x}$ is said to be an *LPV-SS isomorphism* from Σ_1 to Σ_2, if for all $i \in \mathbb{I}_0^{n_p}$

$$A_i^a \mathscr{S} = \mathscr{S} A_i, \; B_i^a = \mathscr{S} B_i, \; C_i^a \mathscr{S} = C_i. \tag{6}$$

In this case Σ_1 and Σ_2 are called *isomorphic* LPV-SS representations. The *order* of Σ, denoted by $\dim(\Sigma)$ is the dimension of its state-space. That is, if Σ is of the form (1), then $\dim(\Sigma) = n_x$. Let f be an input-output map of the form (5). An LPV-SS realization Σ is a *minimal realization of* f, if Σ is a realization of f, and for any LPV-SS representation $\bar{\Sigma}$ which is also a realization of f, $\dim(\Sigma) \leq \dim(\bar{\Sigma})$. We say that Σ is *minimal*, if Σ is a minimal realization of its own input-output map Y_Σ. From [13], it follows that an LPV-SS representation Σ is minimal if and only if it is reachable and observable. In addition, if two minimal LPV-SS realizations are input-output equivalent, then they are isomorphic.

III. MODEL REDUCTION OF LPV-SS REPRESENTATIONS: PRELIMINARIES

In this section, the sub-Markov parameters of a realizable input-output map f and its corresponding LPV-SS representation Σ will be defined, and the moment matching problem for LPV-SS realizations will be stated formally. To this end, we recall the concepts of an *infinite impulse response (IIR)* representation of an input-output map [18] and the concept of sub-Markov parameters.

Consider an LPV-SS representation Σ of the form (1), and consider its input-output map $f = Y_\Sigma$. Recall from [18] that for any input sequence $\mathbf{u} = \{u(k)\}_{k=0}^{\infty}$ and scheduling sequence $\mathbf{p} = \{p(k)\}_{k=0}^{\infty}$,

$$f(\mathbf{u},\mathbf{p})(t) = Y_\Sigma(\mathbf{u},\mathbf{p})(t) = \sum_{m=0}^{t} (h_m \diamond p)(t)u(t-m) \quad (7)$$

for all $t \in \mathbb{N}$ where

$$\begin{aligned}&(h_0 \diamond p)(t) = 0,\ (h_1 \diamond p)(t) = C(p(t))B(p(t-1)),\\ &(h_m \diamond p)(t) = C(p(t))\left(\prod_{l=1}^{m-1} A(p(t-l))\right)B(p(t-m)).\end{aligned} \quad (8)$$

The representation above is called the IIR of $f = Y_\Sigma$. The map f in (7) is absolute convergent for all $\mathbb{P} \in P^{\mathbb{N}}$ if the represented system is IO asymptotically stable. From (8) and (2), it can be seen that the terms $(h_m \diamond p)(t)$, $m \geq 0$ can be written as follows:

$$\begin{aligned}&(h_0 \diamond p)(t) = 0,\ (h_1 \diamond p)(t) = \sum_{q=0}^{n_p}\sum_{q_0=0}^{n_p} C_q B_{q_0} p_q(t) p_{q_0}(t-1),\\ &(h_m \diamond p)(t) =\\ &\sum_{q=0}^{n_p}\sum_{j_1=0}^{n_p}\cdots\sum_{j_{m-1}=0}^{n_p}\sum_{q_0=0}^{n_p} C_q A_{j_1}\cdots A_{j_{m-1}} B_{q_0} \hat{p}_{q j_1 \cdots j_{m-1} q_0},\end{aligned} \quad (9)$$

where $p_0(k) = 1$ for all $k \in \mathbb{I}_0^t$ and $\hat{p}_{q j_1 \cdots j_{m-1} q_0} = p_q(t)p_{j_1}(t-1)\cdots p_{j_{m-1}}(t-m+1)p_{q_0}(t-m)$.

Now we are ready to define the sub-Markov parameters of Σ. To this end, we introduce the symbol ε to denote the empty sequence of integers, i.e., ε will stand for a sequence of length zero and we denote by $\mathscr{S}(\mathbb{I}_0^{n_p})$ the set $\{\varepsilon\} \cup \{j_1 \cdots j_m \mid m \geq 1, j_1, \ldots, j_m \in \mathbb{I}_0^{n_p}\}$ of all sequence of integers from $\mathbb{I}_0^{n_p}$, including the empty sequence. If $s \in \mathscr{S}(\mathbb{I}_0^{n_p})$, then $|s|$ denotes the length of the sequence s. By convention, if $s = \varepsilon$, then $|s| = 0$. The coefficients

$$\begin{aligned}&\eta_{q,q_0}^{\Sigma}(\varepsilon) = C_q B_{q_0},\\ &\eta_{q,q_0}^{\Sigma}(j_1 \cdots j_m) = C_q A_{j_1} \cdots A_{j_m} B_{q_0},\end{aligned} \quad (10)$$

$m \geq 1$; $q, j_1, \ldots, j_m, q_0 \in \mathbb{I}_0^{n_p}$ appearing in (9) are called the *sub-Markov parameters* of the LPV-SS representation Σ. In the sequel, the sub-Markov parameters $\eta_{q,q_0}^{\Sigma}(s)$ with $q, q_0 \in \mathbb{I}_0^{n_p}$, $s \in \mathscr{S}(\mathbb{I}_0^{n_p})$, $|s| = m$, will be called *sub-Markov parameters of* Σ *of length* m. The intuition behind this terminology is as follows: the length of a sub-Markov parameter is determined by the number of A_j matrices which appear in (10) as factors.

Note the sub-Markov parameters do not depend on the particular choice of an LPV-SS representation, but on the choice of the input-output map (provided that we fix an affine depency of the matrices of the LPV-SS representation on the scheduling variable). From [13] it follows that if Σ_1, Σ_2 are two LPV-SS representations with static affine dependence on the scheduling variable, then their input-output maps are equal, if and only if their respective sub-Markov parameters are equal, i.e., $Y_{\Sigma_1} = Y_{\Sigma_2} \iff \forall s \in \mathscr{S}(\mathbb{I}_0^{n_p}) : \eta_{q,q_0}^{\Sigma_1}(s) = \eta_{q,q_0}^{\Sigma_2}(s)$. Note also that another way to interpret the sub-Markov parameters is that they correspond to the derivatives of f with respect to the scheduling variable.

Recall that $p_0(k) = 1$ for all $k \in \mathbb{I}_0^t$. In addition, observe from (8), that the output $y(t)$, for $t \geq 1$ of an LPV-SS representation corresponding to an input sequence $\mathbf{u} = \{u(k)\}_{k=0}^{\infty}$ and a scheduling sequence $\mathbf{p} = \{p(k)\}_{k=0}^{\infty}$ is uniquely determined by the sub-Markov parameters of length up to $t-1$ i.e., only the sub-Markov parameters of length up to $t-1$ appear in the output $y(t)$. Hence, if the sub-Markov parameters of length up to $t-1$ of two LPV-SS representations Σ and $\bar{\Sigma}$ coincide, it means that Σ and $\bar{\Sigma}$ will have the same input-output behavior up to time t for arbitrary input and scheduling sequences. This discussion is formalized below.

Lemma 1 (I/O equivalence and sub-Markov parameters): For any LPV-SS representations Σ_1, Σ_2,

$$\forall(\mathbf{u},\mathbf{p}) \in \mathscr{U} \times \mathscr{P}, k \in \mathbb{I}_0^t : \quad Y_{\Sigma_1}(u,p)(k) = Y_{\Sigma_2}(u,p)(k)$$

if and only if

$$\forall s \in \mathscr{S}(\mathbb{I}_0^{n_p}), q, q_0 \in \mathbb{I}_0^{n_p}, |s| \leq t-1 : \quad \eta_{q,q_0}^{\Sigma_1}(s) = \eta_{q,q_0}^{\Sigma_2}(s)$$

This prompts us to introduce the following definition.

Definition 1: Let Σ be an LPV-SS representation of the form (1). An LPV-SS representation $\bar{\Sigma}$ of the form (3) is called a N-partial realization of $f = Y_\Sigma$, for some $N \in \mathbb{N}$, if

$$\forall s \in \mathscr{S}(\mathbb{I}_0^{n_p}), q, q_0 \in \mathbb{I}_0^{n_p}, |s| \leq N : \eta_{q,q_0}^{\Sigma}(s) = \eta_{q,q_0}^{\bar{\Sigma}}(s). \quad (11)$$

That is, $\bar{\Sigma}$ is an *N-partial realization* of $f = Y_\Sigma$, if the sub-Markov parameters of Y_Σ and $Y_{\bar{\Sigma}}$ up to length N are equal. In other words, $\bar{\Sigma}$ is an N-partial realization of Y_Σ, if

$$\begin{aligned}C_q B_{q_0} &= \bar{C}_q \bar{B}_{q_0},\ \forall q, q_0 \in \mathbb{I}_0^{n_p},\\ C_q A_{j_1} \cdots A_{j_k} B_{q_0} &= \bar{C}_q \bar{A}_{j_1} \cdots \bar{A}_{j_k} \bar{B}_{q_0},\ ,\forall k \in \mathbb{I}_1^N,\\ &\forall q, q_0, j_1, \ldots, j_k \in \mathbb{I}_0^{n_p}.\end{aligned}$$

The problem of model reduction by moment matching for LPV-SS models can now be formulated as follows.

Problem 1: Let Σ be an LPV-SS representation and let $f = Y_\Sigma$ be its input-output map. Fix $N \in \mathbb{N}$. Find another LPV-SS realization $\bar{\Sigma}$ such that $\dim(\bar{\Sigma}) < \dim(\Sigma)$ and $\bar{\Sigma}$ is an N-partial realization of $f = Y_\Sigma$.

In order to explain the intuition behind this definition, we combine [14, Theorem 4] and [13] to derive the following.

Corollary 1: Assume that Σ is a minimal realization of $f = Y_\Sigma$ and N is such that $2\dim(\Sigma) - 1 \leq N$. Then for any

LPV-SS representation $\bar{\Sigma}$ which is an N-partial realization of f, $\bar{\Sigma}$ is also a realization of $f = Y_\Sigma$ and $\dim(\Sigma) \leq \dim(\bar{\Sigma})$.

Remark 1: Corollary 1 implies that there is a tradeoff between the choice of N and the order of Σ. Assume Σ is a minimal realization of $f = Y_\Sigma$. If N is chosen to be too high, namely if it is such that $N \geq 2n_x - 1$, then it will not be possible to find an LPV-SS representation which is an N-partial realization of f and whose order is lower than n_x. In fact, if the model reduction procedure to be presented in the next section is used with any input $N \geq 2n_x - 1$, then the resulting LPV-SS representation $\bar{\Sigma}$ will be a complete realization of $f = Y_\Sigma$. However, the order of $\bar{\Sigma}$ will be the same as the order of Σ (provided that Σ is minimal). This relation between N and n_x gives an a priori idea of how well the input-output map of $\bar{\Sigma}$ approximates that of Σ. More specifically, we can expect the output error $Y_\Sigma - Y_{\bar{\Sigma}}$ to be smaller when N is increased, as long as $N < 2n_x - 1$. This error will be zero for $N \geq 2n_x - 1$, since in this case $\bar{\Sigma}$ will be a complete realization of Y_Σ.

IV. MODEL REDUCTION OF LPV-SS REPRESENTATIONS

In this section, first, the theorems which form the basis of the model reduction by moment matching will be presented. Then, the algorithm itself will be stated. In the sequel, the image (column space) and kernel (null space) of a real matrix M is denoted by $\mathrm{im}(M)$ and $\ker(M)$ respectively. In addition, $\mathrm{rank}(M)$ is the dimension of $\mathrm{im}(M)$. We will start with presenting the following definitions for LPV-SS realizations of the form (1).

Definition 2 (N-partial unobservability space): The N-partial unobservability space $\mathscr{O}_N(\Sigma)$ of Σ is defined inductively as follows:

$$\begin{aligned} \mathscr{O}_0(\Sigma) &= \bigcap_{q \in \mathbb{I}_0^{n_p}} \ker(C_q), \\ \mathscr{O}_N(\Sigma) &= \mathscr{O}_0(\Sigma) \cap \bigcap_{j \in \mathbb{I}_0^{n_p}} \ker(\mathscr{O}_{N-1}(\Sigma) A_j), \; N \geq 1. \end{aligned} \tag{12}$$

From [12], [13], it follows that Σ is observable if and only if $\mathscr{O}_N(\Sigma) = \{0\}$ for all $N \geq n_x - 1$.

Definition 3 (N-partial reachability space): The N-partial reachability space $\mathscr{R}_N(\Sigma)$ of Σ is defined inductively as follows:

$$\begin{aligned} \mathscr{R}_0(\Sigma) &= \mathrm{span} \bigcup_{q_0 \in \mathbb{I}_0^{n_p}} \mathrm{im}(B_{q_0}), \\ \mathscr{R}_N(\Sigma) &= \mathscr{R}_0(\Sigma) + \sum_{j \in \mathbb{I}_0^{n_p}} \mathrm{im}(A_j \mathscr{R}_{N-1}(\Sigma)), \; N \geq 1, \end{aligned} \tag{13}$$

where the summation operator must be interpreted as the Minkowski sum.

Again, from [12], [13], it follows that Σ is span-reachable if and only if $\dim(\mathscr{R}_N(\Sigma)) = n_x$ for all $N \geq n_x - 1$.

Theorem 1: Let $\Sigma = (n_y, n_u, n_x, \{(A_i, B_i, C_i)\}_{i=0}^{n_p})$ be an LPV-SS representation, let $V \in \mathbb{R}^{n_x \times r}$ be a full column rank matrix such that

$$\mathscr{R}_N(\Sigma) = \mathrm{im}(V).$$

If $\bar{\Sigma} = (n_y, n_u, r, \{(\bar{A}_i, \bar{B}_i, \bar{C}_i)\}_{i=0}^{n_p})$ is an LPV-SS representation such that for each $i \in \mathbb{I}_0^{n_p}$, the matrices $\bar{A}_i, \bar{B}_i, \bar{C}_i$ are defined as

$$\bar{A}_i = V^{-1} A_i V, \; \bar{B}_i = V^{-1} B_i, \; \bar{C}_i = C_i V,$$

where V^{-1} is a left inverse of V, then $\bar{\Sigma}$ is an N-partial realization of the input-output map $f = Y_\Sigma$ of Σ.

This theorem follows from [3], [4] using [13]. See [2] for a detailed proof.

Note that the number r is the number of columns in the full column rank matrix V, hence $r \leq n_x$. This fact leads $\bar{\Sigma}$ to be of reduced order if N is sufficiently small, see Corollary 1. Using a dual argument, we can prove the following.

Theorem 2: Let $\Sigma = (n_y, n_u, n_x, \{(A_i, B_i, C_i)\}_{i=0}^{n_p})$ be an LPV-SS representation, and let $W \in \mathbb{R}^{r \times n_x}$ be a full row rank matrix such that

$$\mathscr{O}_N(\Sigma) = \ker(W).$$

Let W^{-1} be any right inverse of W and let

$$\bar{\Sigma} = (n_y, n_u, r, \{(\bar{A}_i, \bar{B}_i, \bar{C}_i)\}_{i=0}^{n_p})$$

be an LPV-SS representation such that for each $i \in \mathbb{I}_0^{n_p}$, the matrices $\bar{A}_i, \bar{B}_i, \bar{C}_i$ are defined as

$$\bar{A}_i = W A_i W^{-1}, \; \bar{B}_i = W B_i, \; \bar{C}_i = C_i W^{-1}.$$

Then $\bar{\Sigma}$ is an N-partial realization of the input-output map $f = Y_\Sigma$ of Σ.

The proof is similar to that of Theorem 1.

Finally, by combining the proofs of Theorem 1 and Theorem 2, we can show the following.

Theorem 3: Let $\Sigma = (n_y, n_u, n_x, \{(A_i, B_i, C_i)\}_{i=0}^{n_p})$ be an LPV-SS representation, and let $V \in \mathbb{R}^{n_x \times r}$ and $W \in \mathbb{R}^{r \times n_x}$ be respectively full column rank and full row rank matrices such that

$$\mathscr{R}_N(\Sigma) = \mathrm{im}(V), \; \mathscr{O}_N(\Sigma) = \ker(W) \text{ and } \mathrm{rank}(WV) = r.$$

If $\bar{\Sigma} = (n_y, n_u, r, \{(\bar{A}_i, \bar{B}_i, \bar{C}_i)\}_{i=1}^{n_p})$ is an LPV-SS representation such that for each $i \in \mathbb{I}_0^{n_p}$, $\bar{A}_i, \bar{B}_i, \bar{C}_i$ are defined as

$$\bar{A}_i = W A_i V (WV)^{-1}, \; \bar{B}_i = W B_i, \; \bar{C}_i = C_i V (WV)^{-1}$$

then $\bar{\Sigma}$ is a $2N$-partial realization of the input-output map $f = Y_\Sigma$ of Σ.

Now, we will present an efficient algorithm of model reduction by moment matching, which computes either an N or $2N$-partial realization $\bar{\Sigma}$ for an f which is realized by an LPV-SS representation Σ. First, we present algorithms for computing the subspaces $\mathscr{R}_N(\Sigma)$ and $\mathscr{O}_N(\Sigma)$. To this end, we will use the following notation: if M is any real matrix, then denote by $\mathbf{orth}(M)$ the matrix U such that U is full column rank, $\mathrm{im}(U) = \mathrm{im}(M)$ and $U^{\mathrm{T}} U = I$. Note that U can easily be computed from M numerically, see for example the Matlab command `orth`.

The methodology for computing $V \in \mathbb{R}^{n_x \times r}$ such that $\mathrm{im}(V) = \mathscr{R}_N(\Sigma)$ is presented in Algorithm 1 below.

By duality, we can use Algorithm 1 to compute a $W \in \mathbb{R}^{r \times n_x}$ such that $\ker(W) = \mathscr{O}_N(\Sigma)$, see Algorithm 2.

Algorithm 1 Calculate a matrix representation of $\mathscr{R}_N(\Sigma)$,
Inputs: $(\{A_i, B_i\}_{i\in\mathbb{I}_0^{n_p}})$ and N
Outputs: $V \in \mathbb{R}^{n_x\times r}$ such that $\text{rank}(V) = r$, $\text{im}(V) = \mathscr{R}_N(\Sigma)$.

$V := U_0$, $U_0 := \textbf{orth}\begin{bmatrix} B_0 & \cdots & B_{n_p}\end{bmatrix}$.
for $k = 1 \dots N$ **do**
  $V := \textbf{orth}(\begin{bmatrix} V & A_0V & A_1V & \cdots & A_{n_p}V\end{bmatrix})$
end for
return V.

Algorithm 2 Calculate a matrix representation of $\mathscr{O}_N(\Sigma)$
Inputs: $\{A_i, C_i\}_{i\in\mathbb{I}_0^{n_p}}$ and N
Output: $W \in \mathbb{R}^{r\times n_x}$, such that $\text{rank}(W) = r$, and $\ker(W) = \mathscr{O}_N(\Sigma)$.

Apply Algorithm 1 with inputs $(\{A_i^{\mathrm{T}}, C_i^{\mathrm{T}}\}_{i\in\mathbb{I}_0^{n_p}})$ to obtain a matrix V.
return $W = V^{\mathrm{T}}$.

Notice that the computational complexity of Algorithm 1 and Algorithm 2 is polynomial in N and n_x, even though the spaces of $\mathscr{R}_N(\Sigma)$ (resp. $\mathscr{O}_N(\Sigma)$) are generated by images (resp. kernels) of exponentially many matrices. Using Algorithms 1 and 2, we can formulate a model reduction algorithm, see Algorithm 3.

Theorems 1 – 3 imply the correctness of Algorithm 3.

Remark 2 (Minimization of LPV-SS representations): From [13], it follows that if $N \geq n_x - 1$ then

$$\mathscr{R}_N(\Sigma) = \sum_{i=0}^{\infty} \mathscr{R}_i(\Sigma) = \text{span}\{X_\Sigma(\mathbf{u}, \mathbf{p})(t) \mid (\mathbf{u}, \mathbf{p}) \in \mathscr{U} \times \mathscr{P}, t \geq 0\},$$

$$\mathscr{O}_N(\Sigma) = \bigcap_{i=0}^{\infty} \mathscr{O}_i(\Sigma) = \{x \in \mathbb{R}^{n_x} \mid Y_{\Sigma,x}(\mathbf{u}, \mathbf{p})(t) = 0, \forall (\mathbf{u}, \mathbf{p}) \in \mathscr{U} \times \mathscr{P}, \forall t \geq 0\}.$$

In other words, an LPV-SS representation Σ of the form (1) is reachable if and only if the dimension of its N-partial reachability space $\mathscr{R}_N(\Sigma)$ is n_x for all $N \geq n_x - 1$, and Σ is observable if and only if the dimension of its N-partial unobservability space $\mathscr{O}_N(\Sigma)$ is 0 for all $N \geq n_x - 1$. In addition from [13], it follows that Σ is a minimal realization of its own input-output map Y_Σ if and only if Σ is reachable and observable. Hence, using this fact and [12], [18], it can be shown that Algorithm 3 can be used as an order minimization algorithm. That is, Algorithm 3 can be used consecutively with the inputs $N \geq n_x - 1$, $\texttt{Mode} = \texttt{R}$ (in this case, the resulting $\bar{\Sigma}$ will be reachable and it will be a realization of $f = Y_\Sigma$) and $N \geq n_x - 1$, $\texttt{Mode} = \texttt{O}$ (in this case, the resulting $\bar{\Sigma}$ will be observable and it will be a realization of $f = Y_\Sigma$) for reachability and observability reduction for Σ, respectively. In turn, the resulting representation $\bar{\Sigma}$ will be a minimal realization of $f = Y_\Sigma$.

Algorithm 3 Moment matching for LPV-SS representations
Inputs: $\Sigma = (n_y, n_u, n_x, \{(A_i, B_i, C_i)\}_{i=0}^{n_p})$, $\texttt{Mode} \in \{\texttt{R}, \texttt{O}, \texttt{T}\}$ and $N \in \mathbb{N}$.
Output: $\bar{\Sigma} = (n_y, n_u, r, \{(\bar{A}_i, \bar{B}_i, \bar{C}_i)\}_{i=0}^{n_p})$.

Using Algorithm 1-2, compute matrices V and W such that V is full column rank, W is full row rank and $\text{im}(V) = \mathscr{R}_N(\Sigma)$, $\ker(W) = \mathscr{O}_N(\Sigma)$.
if $\text{rank}(V) = \text{rank}(W) = \text{rank}(WV)$ and $\texttt{Mode} = \texttt{T}$ **then**
  Let $r = \text{rank}(V)$ and

$$\bar{A}_i = WA_iV(WV)^{-1},\ \bar{C}_i = C_iV(WV)^{-1},\ \bar{B}_i = WB_i.$$

end if
if $\texttt{Mode} = \texttt{R}$ **then**
  Let $r = \text{rank}(V)$, V^{-1} be a left inverse of V and set

$$\bar{A}_i = V^{-1}A_iV,\ \bar{C}_i = C_iV,\ \bar{B}_i = V^{-1}B_i.$$

end if
if $\texttt{Mode} = \texttt{O}$ **then**
  Let $r = \text{rank}(W)$ and let W^{-1} be a right inverse of W. Set

$$\bar{A}_i = WA_iW^{-1},\ \bar{C}_i = C_iW^{-1},\ \bar{B}_i = WB_i.$$

end if
return $\bar{\Sigma} = (n_y, n_u, r, \{(\bar{A}_i, \bar{B}_i, \bar{C}_i)\}_{i=0}^{n_p})$.

V. NUMERICAL EXAMPLES

In this section, the method stated in the present paper is applied to Example 4 in [18] and the result is compared with the one given in [18]. For this, both procedures are implemented in MATLAB. The codes and the data used for both examples in this section are available from https://kom.aau.dk/~mertb/.

The algorithm is applied to get a 3rd order approximation to the LPV-SS realization of order 4 in Example 4, [18]. The original LPV-SS representation used in this case is of the form $\Sigma = (n_y, n_u, n_x, \{(A_i, B_i, C_i)\}_{i=0}^{n_p})$ with $n_y = n_u = 1$, $n_x = 4$ and $n_p = 3$. When N is chosen to be 1 and $\texttt{Mode} = \texttt{Reach}$, the resulting reduced order model $\bar{\Sigma}$ is a 1-partial realization of Y_Σ of order 3. The scheduling signal used for simulation is of the form $p(t) = \begin{bmatrix} \hat{p} & \sqrt{-\hat{p}} & \sin(\hat{p})\end{bmatrix}^{\mathrm{T}}$ where the parameter $\hat{p}$ takes its values randomly at each time instant, in the interval $[-2\pi, 0]$. In addition, a white input $u(t) \sim \mathscr{N}(0,1)$ is used. The upper limit of the simulation time interval is chosen to be $N + 50 = 51$. Since $N = 1$, the sub-Markov parameters of length at most 1 are matched with the original LPV-SS model Σ. The precise number of matched sub-Markov parameters is thus $(n_p+1)\left(\frac{(n_p+1)^{N+1}-1}{n_p}\right)(n_p+1) = 80$. The original model Σ and the the reduced order model $\bar{\Sigma}$ are simulated for 500 different scheduling and input signal sequences of the type explained above, and their outputs $y(t)$ and $\bar{y}(t)$ are compared for $t = 0, 1, \dots, K$, where K is the number of steps of the simulation. For each simulation, the responses of Σ and $\bar{\Sigma}$ are compared with the best fit rate (BFR) (see [10], [18]) which is defined as

$$\text{BFR} = 100\% \max\left(1 - \frac{\sqrt{\sum_{t=0}^{K}\|y(t) - \bar{y}(t)\|_2^2}}{\sqrt{\sum_{t=0}^{K}\|y(t) - y_m\|_2^2}}, 0\right)$$

TABLE I

COMPARISON OF ALG. 3 AND THE ALG. IN [18]

The Proc.	Mean BFR	Best BFR	Worst BFR	Run Time
Alg. 3	76.5710%	86.5821%	64.9409%	0.0430 s
Alg. in [18]	75.4364%	85.4157%	58.5798%	0.0711 s

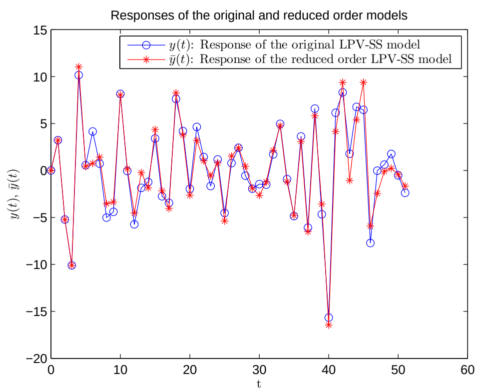

Fig. 1. The responses of the original LPV-SS model Σ of order 4 and the reduced order LPV-SS model $\bar{\Sigma}$ of order 3 acquired by Algorithm 3. The BFR for this simulation is $= 76.5773\%$.

where y_m is the mean of $\{y(t)\}_{t=0}^{K}$.

For this example, the algorithms stated in this paper and in [18] are implemented for comparison. The mean of the BFRs, which is computed over 500 simulations, can be seen on Table I. In addition the best and worst BFRs over 500 simulations and the run-times for one single reduction algorithm are also shown in Table I. The outputs $y(t)$ and $\bar{y}(t)$ of the simulation which give the closest value to the mean of the BFRs are shown in Fig. 1. We used Algorithm 3 to perform model reduction using moment matching. From Table I, it can be seen that both algorithms result in almost the same fit rates, whereas the algorithm stated in the present paper provides a 50% reduction in terms of computational complexity.

Note that LPV-SS examples with much bigger order n and scheduling space dimension n_p are available for freely experimenting on https://kom.aau.dk/~mertb/. The present example is chosen for comparison with the method in [18] (the same example is used in [18]) and for its simplicity. See also [2] for a detailed example where the method in [18] breaks down due to memory limitations, whereas the present method functions successfully.

VI. CONCLUSIONS

A model reduction method is presented for discrete time LPV-SS representations with affine static dependence on the scheduling variable. The method makes it possible to find a reduced order approximation to the original LPV-SS model, which has the same input-output behavior for scheduling and input sequences of a pre-defined, limited length. The presented method can also be used for reachability and observability reduction (i.e., minimization) for LPV-SS models.

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